

# A REMARK ON REGULARITY

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*To the memory of our friend Tony Geramita*

**ABSTRACT.** We show that the regularity and related invariants of products of powers of ideals are linear in the exponents if these are large enough, provided that each ideal is generated by elements of constant degree. A counterexample shows that linearity is false without the condition, but the regularity is always given by the maximum of finitely many affine-linear polynomials.

## 1. INTRODUCTION

Let  $I$  be an homogeneous ideal of a polynomial ring  $R = K[x_1, \dots, x_n]$ . Cutkosky, Herzog and Trung [6] and, independently, Kodiyalam [9] proved in that the Castelnuovo-Mumford regularity  $\text{reg}(I^a)$  of the powers of  $I$  is a linear function of  $a \in \mathbb{N}$  for large  $a$ . In [6, Remark pg.252] it is asserted that the same result holds for products of powers of ideals  $I_1, \dots, I_m$ , i.e.,  $\text{reg}(I_1^{a_1} \cdots I_m^{a_m})$  is a linear function in  $a = (a_1, \dots, a_m) \in \mathbb{N}^m$  if  $a_i \gg 0$  for every  $i$ . We show that this is actually the case when each ideal  $I_i$  is generated by polynomials of a given degree, but false in general.

We refer the reader to [4] for basic commutative algebra.

## 2. REGULARITY FOR POWERS AND PRODUCTS

For a finitely generated graded non-zero  $R$ -module  $M$  and a nonnegative integer  $j \leq \text{pd}_R(M)$  we denote the largest degree of a minimal generator of the  $j$ -th syzygy module by  $t_j^R(M)$ , and, by convention, set  $t_j^R(M) = -\infty$  if  $j > \text{pd}_R(M)$ . By definition one has  $\text{reg}_R(M) = \max\{t_j^R(M) - j : j \geq 0\}$ .

Let  $I_1, \dots, I_m$  be non-zero homogeneous ideals of  $R$  and let  $f_{i1}, \dots, f_{ig_i}$  be a minimal homogeneous generating system of  $I_i$ . Set  $d_{ij} = \deg f_{ij}$  and

$$B = R[z_{ij} : 1 \leq i \leq m, 1 \leq j \leq g_i] = K[x_1, \dots, x_n, z_{ij} : 1 \leq i \leq m, 1 \leq j \leq g_i].$$

To simplify the exposition we set

$$I^a = I_1^{a_1} \cdots I_m^{a_m} \quad \text{for } a \in \mathbb{N}^m,$$

and we say that a formula (or a property) holds for  $a \gg 0$  if there exist a  $b \in \mathbb{N}^m$  such that it holds for every  $a \in b + \mathbb{N}^m$ .

The multigraded Rees algebra  $R(I_1, \dots, I_m)$  can be seen as the subalgebra of  $R[t_1, \dots, t_m]$  whose elements have the form  $\sum_{a \in \mathbb{N}^m} F_a t^a$  with  $F_a \in I^a$ . The polynomial ring  $R[t_1, \dots, t_m]$  is naturally  $\mathbb{Z} \times \mathbb{Z}^m$ -graded if we extend the  $\mathbb{Z}$ -grading on  $R$  by setting  $\deg(x_i) = (1, 0)$

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and  $\deg t_j = (0, e_j)$  where  $e_j$  denotes the  $j$ -th unit vector in  $\mathbb{Z}^m$ . Evidently  $R(I_1, \dots, I_m)$  is a graded subalgebra. It has a structure of  $B$ -module via the  $R$ -algebra map sending  $z_{ij}$  to  $f_{ij}t_i$ . This map is  $\mathbb{Z} \times \mathbb{Z}^m$ -graded if we equip  $B$  with the graded structure induced by the assignment  $\deg(x_i) = (1, 0)$  and  $\deg(z_{ij}) = (d_{ij}, e_i)$ .

Let  $A = K[z_{ij} : 1 \leq i \leq m, 1 \leq j \leq g_i] \subset B$  with the induced  $\mathbb{Z} \times \mathbb{Z}^m$ -graded structure. Note that  $A$  has no elements of degree  $(g, 0) \in \mathbb{Z} \times \mathbb{Z}^m$  such that  $g \neq 0$ . Therefore, given a finitely generated  $\mathbb{Z} \times \mathbb{Z}^m$ -graded  $A$ -module  $M$ , the maximum  $\mathbb{Z}$ -degree in an  $a$ -graded component,

$$\rho_M(a) = \sup\{i \in \mathbb{Z} : M_{(i,a)} \neq 0\}, \quad a \in \mathbb{Z}^m,$$

is finite. The key to the result about the behavior of regularity is a description of  $\rho_M(a)$ : it is the supremum over the values of finitely many affine-linear polynomials  $L_k(a)$  where each  $L_k$  is defined on a certain subset of  $\mathbb{Z}^m$ . In order to describe these subsets we set  $[m] = \{1, \dots, m\}$ , and

$$\text{supp } L = \{i \in [m] : \alpha_i \neq 0\} \subset [m]$$

for a affine-linear polynomial  $L = \sum_{i=1}^m \alpha_i u_i + \alpha_0$ . The nonzero coefficients of the relevant affine-linear polynomials are given by  $\mathbb{Z}$ -degrees  $d_{ij}$  of the indeterminates of  $A$ :

**Lemma 2.1.** *Let  $M$  be a finitely generated  $\mathbb{Z} \times \mathbb{Z}^m$ -graded  $A$ -module. Then there exist  $w_1, \dots, w_c \in \mathbb{Z}^m$  and affine-linear polynomials  $L_1, \dots, L_c$  on  $\mathbb{Z}^m$ ,*

$$L_k(u) = \sum_{i=1}^m \lambda_{ki} u_i + \lambda_{k0}, \quad \lambda_{k0} \in \mathbb{Z}, \quad \lambda_{ki} \in \{0\} \cup \{d_{i1}, \dots, d_{ig_i}\}, \quad i = 1, \dots, m$$

such that

$$\rho_M(a) = \sup \left\{ L_k(a) : 1 \leq k \leq c \text{ and } a \in w_k + \sum_{i \in \text{supp } L_k} \mathbb{N} e_i \right\}$$

for every  $a \in \mathbb{Z}^m$ . In particular we have

$$\rho_M(a) = \sup \{ L_k(a) : 1 \leq k \leq c \text{ and } \text{supp } L_k = [m] \} \quad \text{for } a \gg 0.$$

*Proof.* We represent  $M$  as a quotient  $F/U$  of the graded free module  $F$  by a graded submodule  $U$ . Then we replace  $U$  with its initial submodule  $\text{in}(U)$  with respect to some term order. Next we filter  $F/\text{in}(U)$  so that the successive quotients are shifted copies of quotients of  $A$  by monomial prime ideals of  $A$ . It follows that the (multigraded) Hilbert series

$$\text{HS}_M(x, t) = \sum_{(i,a) \in \mathbb{Z} \times \mathbb{Z}^m} \dim M_{(i,a)} x^i t^a \in \mathbb{Q}[[x, t_1, \dots, t_m]]$$

can be written as the sum of the Hilbert series of residue class rings  $A/P_k(-v_k, -w_k)$  where  $P_k$  an ideal generated by a subset of the variables of  $A$  and  $(-v_k, -w_k) \in \mathbb{Z} \times \mathbb{Z}^m$  and is a shift. That is,

$$\text{HS}_M(t, x) = \sum_{i=1}^c x^{v_k} t^{w_k} \text{HS}_{A/P_k}(x, t)$$

and therefore

$$\rho_M(a) = \sup \{ \rho_{A/P_k}(a - w_k) + v_k : k = 1, \dots, c \}.$$

Set

$$S_k = \{i : \text{there exists } j \text{ such that } z_{ij} \notin P_k\}$$

and

$$\lambda_{ki} = \max\{d_{ij} : z_{ij} \notin P_k\}$$

if  $i \in S_k$  and  $\lambda_{ki} = 0$  otherwise. Since

$$\text{HS}_{A/P_k}(x, t) = \frac{1}{\prod_{z_{ij} \notin P_k} (1 - x^{d_{ij}} t_i)}$$

we have

$$\rho_{A/P_k}(a) = \begin{cases} \sum_{j=1}^m \lambda_{ki} a_i & \text{if } a \in \sum_{i \in S_k} \mathbb{N} e_i, \\ -\infty & \text{otherwise.} \end{cases}$$

Summing up, we obtain the desired description where  $L_k$  is defined as

$$L_k(u) = \sum_{i=1}^m \lambda_{ki} (u_i - w_{ki}) + v_k.$$

□

Set  $\mathfrak{m} = (x_1, \dots, x_n)$  and  $\mathcal{R} = R(I_1, \dots, I_m)$ . Consider the Koszul complex  $K(\mathfrak{m}, \mathcal{R})$  and its homology  $H(\mathfrak{m}, \mathcal{R})$ . By construction  $H_j = H_j(\mathfrak{m}, \mathcal{R})$  is a finitely generated  $\mathbb{Z} \times \mathbb{Z}^m$  graded  $B$ -module annihilated by  $\mathfrak{m}$  and  $\dim_K H_j(\mathfrak{m}, \mathcal{R})_{(u,a)} = \beta_{ju}(I^a)$ . Hence  $H_j$  is a finitely generated  $\mathbb{Z} \times \mathbb{Z}^m$  graded  $A$ -module and

$$\rho_{H_j}(a) = t_j^R(I^a).$$

Then Lemma 2.1 applies so that  $t_j^R(I^a)$  is given as the supremum of finitely many affine-linear polynomials each of which is evaluated on a translated (possibly lower dimensional) non-negative orthant. Indeed, one can explicitly compute such a representation by following the steps described in the proof of 2.1 and applying them to the Koszul homology module  $H_i$ .

For simplicity we state only the main consequence about the asymptotic behavior of  $t_i(I^a)$  and  $\text{reg}(I^a)$ . It is of course determined by the affine-linear polynomials  $L_k$  with  $\text{supp } L_k = [m]$ .

**Theorem 2.2.** *Let  $I_1, \dots, I_m$  be given as above. Then, for every  $0 \leq j \leq n-1$ , there exist  $b_j \in \mathbb{N}$  and affine-linear polynomials  $L_k^{(j)}(u) = \sum_{i=1}^m \lambda_{ki}^{(j)} u_i + \lambda_{k0}^{(j)}$  with  $1 \leq k \leq b_j$  such that  $\lambda_{ki}^{(j)} \in \{d_{i1}, \dots, d_{ig_i}\}$  and  $\lambda_{k0}^{(j)} \in \mathbb{Z}$  and such that*

$$t_j^R(I^a) = \sup\{L_k^{(j)}(a) : 1 \leq k \leq b_j\} \quad \text{for } a \gg 0.$$

*In particular,  $\text{pd}_R(I^a)$  is constant for  $a \gg 0$  and*

$$\text{reg}(I^a) = \sup\{L_k^{(j)}(a) - j : 0 \leq j \leq n-1 \text{ and } 1 \leq k \leq b_j\} \quad \text{for } a \gg 0.$$

If  $I_i$  is generated in a single degree, say  $d_i$ , then there is only one choice for  $\lambda_{ki}^{(j)}$ ,  $i > 0$ , namely  $d_i$ . If this holds for all ideals  $I_i$ , then the supremum is obviously given by a single affine-linear polynomial, and we obtain

**Corollary 2.3.** *If each  $I_i$  is generated in a single degree, say  $d_i$ , then  $\mathrm{pd}_R(I^a)$  is constant for  $a \gg 0$ , say equal to  $p$ , and for each  $j$ ,  $0 \leq j \leq p$ , there exists a affine-linear polynomial  $L^{(j)}(u) = \sum_{i=1}^m d_i u_i + \lambda_0^{(j)}$  with  $\lambda_0^{(j)} \in \mathbb{Z}$  such that*

$$t_j^R(I^a) = L^{(j)}(a) \quad \text{for } a \gg 0.$$

*In particular, there exists an affine-linear polynomial  $L(u) = \sum_{i=1}^m d_i u_i + \lambda_0$  such that*

$$\mathrm{reg}(I^a) = L(a) \quad \text{for } a \gg 0.$$

The conclusion of the corollary holds for a single ideal without any hypothesis on the degrees of the generators since the supremum of finitely many affine-linear polynomials on  $\mathbb{Z}$  is always given by a single one for large values of the argument.

**Remark 2.4.** Let  $J_i$  be a homogeneous reduction of  $I_i$  for every  $i$ . Then  $\mathcal{R} = R(I_1, \dots, I_m)$  is a finitely generated algebra over  $R(J_1, \dots, J_m)$ . It follows that the Koszul homology  $H(x, \mathcal{R})$  is finitely generated over a polynomial ring whose variables have degree  $(v, e_i) \in \mathbb{Z} \times \mathbb{Z}^m$  where  $v$  varies in the set of degrees of the generators of  $J_i$ . Hence the coefficients  $\lambda_{ki}^{(j)}$  in Theorem 2.2 can be taken in the set of the degrees of the generators of  $J_i$ . Therefore Corollary 2.3 holds more generally if each  $I_i$  has a reduction generated in a single degree. Note that Kodiyalam proved in [9] that for a single ideal  $I_1$  one has  $\mathrm{reg}(I_1^a) = ad + b$  for  $a \in \mathbb{N}$  and  $a \gg 0$  where  $d$  is the minimum of  $t_0(J)$  where  $J$  varies in the set of the homogeneous reduction of  $I_1$ .

**Remark 2.5.** Under the assumption that  $I_i$  is generated in degree  $d_i$ , in [3] we have observed that  $\mathrm{reg}(I^a) \leq \sum_{i=1}^m d_i a_i + \mathrm{reg}_x \mathcal{R}$  for all  $a \in \mathbb{N}^m$ . So the constant  $\lambda_0$  in 2.3 is  $\leq \mathrm{reg}_x \mathcal{R}$ . But in general the inequality can be strict. For example, already for  $m = 1$  if  $I_1$  is an ideal such that all the powers but the first have a linear resolution, then  $\lambda_0 = 0$  and  $\mathrm{reg}_x \mathcal{R} > 0$ . Examples of ideals of this kind or making other shorts of regularity jumps are described in [2] and [5].

**Remark 2.6.** So far we have considered ideals in polynomial rings. If  $R$  is not a polynomial ring (but still standard graded), then the projective dimension and the regularity of  $R$ -modules can be infinite. Nevertheless, one has a bound for  $t_j^R(I^a)$  for all  $j$  as in Theorem 2.2. Instead of the Koszul complex, the resolution of  $K$  over the polynomial ring, one must use the free resolution of  $K$  over  $R$  (of infinite length if  $R$  is not a polynomial ring).

**Remark 2.7.** In [7] the author provides a linear upper bound for  $\mathrm{reg}(I^a)$  where  $I_1, \dots, I_m$  are ideals in a standard graded algebra  $R$  over an Artinian local ring. Here the regularity is computed with respect to the local cohomology modules supported on the positive components of  $R$ .

When  $R$  is a standard graded polynomial ring over a field, then our results apply since the regularity based on local cohomology coincides with the “resolution” regularity.

### 3. EXAMPLES

We now discuss an example showing that, in general,  $t_j^R(I^a)$  and  $\mathrm{reg}(I^a)$  need not be linear functions of  $a$  for  $a \gg 0$ .

**Example 3.1.** Consider the ideals  $I_1 = (x, y^2)$ ,  $I_2 = (x^2, y)$  and in the polynomial ring  $R = K[x, y]$ . One has

$$t_i(I_1^{a_1} I_2^{a_2}) = \begin{cases} \max\{2a_1 + a_2, a_1 + 2a_2\} + i & \text{if } i = 0, 1 \text{ and } (a_1, a_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}, \\ 0 & \text{if } i = 0 \text{ and } (a_1, a_2) = (0, 0). \end{cases}$$

So that

$$\text{reg}(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2, a_1 + 2a_2\} \text{ for all } (a_1, a_2) \in \mathbb{N}^2.$$

To establish these formulas we propose two approaches. First we follow the strategy outlined above and deduce the result from the study of the multigraded Hilbert series of the Koszul homology modules  $H_i(x, y, R(I_1, I_2))$ . Secondly we write down the resolution of  $I_1^{a_1} I_2^{a_2}$  directly.

The presentation of the Rees algebra  $\mathcal{R} = R(I_1, I_2)$  is given by:

$$B/P \simeq \mathcal{R}, \quad B = R[z_{11}, z_{12}, z_{21}, z_{22}],$$

with  $\mathbb{Z} \times \mathbb{Z}^2$  graded structure induced by  $\deg(x) = \deg(y) = (1, 0, 0)$ ,  $\deg(z_{11}) = (1, e_1)$ ,  $\deg(z_{12}) = (2, e_1)$ ,  $\deg(z_{21}) = (2, e_2)$ ,  $\deg(z_{22}) = (1, e_2)$ . Here

$$P = (y^2 z_{11} - x z_{12}, y z_{21} - x^2 z_{22}, x y z_{11} z_{22} - z_{12} z_{21}).$$

Set  $A = K[z_{11}, z_{12}, z_{21}, z_{22}]$  so that  $H_0 = H_0(x, y, \mathcal{R}) = \mathcal{R}/(x, y) = A/(z_{12} z_{21})$ . Using a filtration or by direct computation one has:

$$\text{HS}_{H_0}(x, t_1, t_2) = \frac{1}{(1 - x t_1)(1 - x^2 t_1)(1 - x t_2)} + \frac{x^2 t_2}{(1 - x t_1)(1 - x t_2)(1 - x^2 t_2)}. \quad (3.1)$$

From this we deduce that

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2 : a \in \mathbb{N}^2, a_1 + 2a_2 : a \in e_2 + \mathbb{N}^2\},$$

that is,

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2, a_1 + 2a_2\} \text{ for all } a \in \mathbb{N}^2.$$

Similarly for  $H_1 = H_1(x, y, \mathcal{R})$  one obtains:

$$\text{HS}_{H_1}(x, t) = \frac{x^3 t_1}{(1 - x t_1)(1 - x^2 t_1)(1 - x t_2)} + \frac{x^3 t_2}{(1 - x t_1)(1 - x^2 t_2)(1 - x t_2)}. \quad (3.2)$$

Then it follows that

$$t_1(I_1^{a_1} I_2^{a_2}) = \sup\{2a_1 + a_2 + 1 : (a_1, a_2) \in e_1 + \mathbb{N}^2, a_1 + 2a_2 + 1 : (a_1, a_2) \in e_2 + \mathbb{N}^2\}.$$

Hence

$$t_1(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2 + 1, a_1 + 2a_2 + 1\}$$

for all  $a \in \mathbb{N}^2 \setminus \{(0, 0)\}$ .

Let us now sketch the approach via free resolutions. First one checks that  $I_1^{a_1} I_2^{a_2}$  is minimally generated by the  $1 + a_1 + a_2$  elements

$$\begin{array}{c} x^{a_1} y^{a_2}, \quad x^{a_1-1} y^{a_2+2}, x^{a_1-2} y^{a_2+4}, \dots, y^{a_2+2a_1}, \\ \parallel \\ x^{a_1} y^{a_2}, \quad x^{a_1+2} y^{a_2-1}, x^{a_1+4} y^{a_2-2}, \dots, x^{a_1+2a_2}. \end{array}$$

This confirms our formula

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2, a_1 + 2a_2\} \quad \text{for all } a \in \mathbb{N}^2.$$

By Hilbert's syzygy theorem  $I_1^{a_1} I_2^{a_2}$  has a free resolution of length 1 (if  $a_1 + a_2 > 0$ ). We claim the syzygy module of  $I_1^{a_1} I_2^{a_2}$  is generated by the rows of the following  $(a_1 + a_2) \times (a_1 + a_2 + 1)$  matrix

$$\varphi = \begin{pmatrix} y^2 & -x & & & & & & & \\ & y^2 & -x & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & y^2 & -x & & & & \\ x^2 & & & & & -y & & & \\ & & & & x^2 & -y & & & \\ & & & & & x^2 & -y & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & x^2 & -y \end{pmatrix}$$

In fact, the rows of  $\varphi$  are syzygies. Therefore we have a complex

$$0 \rightarrow R^{a_1+a_2} \xrightarrow{\varphi} R^{1+a_1+a_2} \rightarrow I_1^{a_1} I_2^{a_2} \rightarrow 0.$$

Both  $x^{a_1+2a_2}$  and  $y^{2a_1+a_2}$  appear as maximal minors of  $\varphi$ . Thus the ideal of maximal minors of  $\varphi$  has grade 2 in  $R$ . Now the Buchsbaum-Eisenbud exactness criterion (for example, see [4, 1.4.3]) implies that our complex is exact, and so we know the first syzygy module of  $I_1^{a_1} I_2^{a_2}$ . Keeping track of the degrees one obtains that

$$t_1(I_1^{a_1} I_2^{a_2}) = \max\{2a_1 + a_2 + 1, a_1 + 2a_2 + 1\}$$

for all  $a \in \mathbb{N}^2 \setminus \{(0,0)\}$ .

**Example 3.2.** In the example above the regularity and the invariants  $t_i(I^a)$  are given by the supremum of two affine-linear polynomials for  $a \gg 0$ . But even for  $m = 2$  the number of affine-linear polynomials can be larger. An example is given by the ideals  $I_1 = (x, y^2, z^3)$  and  $I_2 = (x^4, y^3, z)$  in  $R = K[x, y, z]$  for which each  $t_i(I_1^{a_1} I_2^{a_2})$  and the regularity is given asymptotically by the maximum of 3 affine-linear polynomials. More precisely for every  $a \geq (1, 1)$  one has:

$$t_0(I_1^{a_1} I_2^{a_2}) = \max\{a_1 + 4a_2 + 1, \quad 2a_1 + 3a_2, \quad 3a_1 + a_2 \quad\},$$

$$t_1(I_1^{a_1} I_2^{a_2}) = \max\{a_1 + 4a_2 + 2, \quad 2a_1 + 3a_2 + 1, \quad 3a_1 + a_2 + 2 \quad\},$$

$$t_2(I_1^{a_1} I_2^{a_2}) = \max\{a_1 + 4a_2 + 3, \quad 2a_1 + 3a_2 + 2, \quad 3a_1 + a_2 + 3 \quad\},$$

$$\text{reg}(I_1^{a_1} I_2^{a_2}) = \max\{a_1 + 4a_2 + 1, \quad 2a_1 + 3a_2, \quad 3a_1 + a_2 + 1 \quad\}.$$

The computation has been done following the strategy outlined above, making use of Macaulay 2 [8] and CoCoA [1] for the computation of the Hilbert series of the Koszul homology  $H_i(x, y, z, R(I_1, I_2))$  and of the filtration allowing to decompose the Hilbert series

as a sum of rational series with positive numerators and “partial” denominators similarly to what we have done in (3.1) and (3.2).

We expect that there is no bound on the number of affine-linear polynomials, at least if one allows an arbitrary number of indeterminates.

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